

# ON CURVATURE AND HYPERBOLICITY OF MONOTONE HAMILTONIAN SYSTEMS

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**ABSTRACT.** Assume that a Hamiltonian system is monotone. In this paper, we give several characterizations on when such a system is Anosov. Assuming that a monotone Hamiltonian system has no conjugate point, we show that there are two distributions which are invariant under the Hamiltonian flow. We show that a monotone Hamiltonian flow without conjugate point is Anosov if and only if these distributions are transversal. We also show that if the reduced curvature of the Hamiltonian system is non-positive, then the flow is Anosov if and only if the reduced curvature is negative somewhere along each trajectory. This generalizes the corresponding results on geodesic flows in [10].

## 1. INTRODUCTION

In this paper, we consider when a Hamiltonian system is Anosov. Let us first recall the definition of a Anosov flow. Let  $X$  be a vector field defined on a manifold  $N$ . Its flow is Anosov if there is a Riemannian metric  $\langle \cdot, \cdot \rangle$  and a splitting  $TN = \mathbb{R}X \oplus \Delta^+ \oplus \Delta^-$  of the tangent bundle  $TN$  of  $N$  such that the followings hold.

- (1)  $\Delta^\pm$  are distributions which are invariant under the flow  $\varphi_t$  of  $X$ ,
- (2) there are positive constants  $c_1$  and  $c_2$  such that  $|d\varphi_{\pm t}(v)| \leq c_1 e^{-c_2 t} |v|$  for all  $v$  in  $\Delta^\pm$  and for all  $t \geq 0$ .

In [6], it was shown that the geodesic flow on the unit sphere bundle of a compact manifold is Anosov if the sectional curvature of the manifold is everywhere negative. This result was generalized to monotone Hamiltonian systems in [3] using the curvature invariants introduced in [2]. On the other hand, it was shown in [10] that there are many alternative characterizations of Anosov geodesic flow under the assumption that the flow has no conjugate point. Some of them were extended by [9] to Hamiltonian systems arising from the classical action functionals in calculus of variations.

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In this paper, we extend the results in [10, 9] to monotone Hamiltonian systems. Let us first recall the definition and the setup of a monotone Hamiltonian system. Let  $M$  be a manifold equipped with a symplectic structure  $\omega$  and a Lagrangian distribution  $\Lambda$ . Let  $H : M \rightarrow \mathbb{R}$  be a fixed Hamiltonian and let us denote the corresponding Hamiltonian vector field by  $\vec{H}$ . Recall that  $\vec{H}$  is defined by  $\omega(\vec{H}, \cdot) = -dH(\cdot)$ . Let  $V_1$  and  $V_2$  be two sections of  $\Lambda$  and let  $\langle V_1, V_2 \rangle$  be defined by

$$\langle V_1, V_2 \rangle = \omega([\vec{H}, V_1], V_2).$$

on  $\Lambda$ . It is not hard to see that  $\langle \cdot, \cdot \rangle$  defines an inner product on the distribution  $\Lambda$ . The Hamiltonian vector field  $\vec{H}$  is monotone if  $\langle \cdot, \cdot \rangle$  defines a Riemannian metric on  $\Lambda$ .

The monotonicity of a Hamiltonian  $H$  means essentially that the restriction of  $H$  to each space  $\Lambda_\alpha$  is strictly convex. More precisely, let  $H : T^*N \rightarrow \mathbb{R}$  be a Hamiltonian defined on the cotangent bundle  $M = T^*N$  of a manifold  $N$ . Assume that the Hamiltonian is fibrewise strictly convex. That is  $H|_{T^*N_x}$  is strictly convex for each  $x$  in  $N$ . Then the Hamiltonian vector field  $\vec{H}$  is monotone if  $T^*N$  is equipped with the symplectic structure  $\omega = d\theta$ , where  $\theta$  is the tautological one form defined by  $\theta_\alpha(V) = \alpha(d\pi(V))$ . These are the Hamiltonians considered in [9]. More generally, one can consider twisted symplectic structure defined on  $T^*N$  by  $\omega = d\theta + \pi^*\eta$ , where  $\eta$  is any closed two form on  $N$ . Then the Hamiltonian vector field is monotone with respect to this twisted symplectic structure if and only if the Hamiltonian is fibrewise strictly convex. Note also that if  $\vec{H}$  is monotone with respect to a symplectic structure  $\omega$  and a Lagrangian distribution  $\Lambda$ , then we can slightly perturb the structures  $(\omega, \Lambda)$  and  $\vec{H}$  is still monotone.

Let  $\alpha$  be a point in  $M$ . We say that the Hamiltonian flow  $\varphi_t$  of  $\vec{H}$  has no point conjugate to  $\alpha$  if  $d\varphi_t(\Lambda_\alpha)$  intersects transversely with  $\Lambda_{\varphi_t(\alpha)}$  for all time  $t$ . If the Hamiltonian flow has no conjugate point, then there are two (measurable) distributions which are invariant under the Hamiltonian flow. This was first proved, in the case of the geodesic flow, in [14] (see also [9] for an extension).

**Theorem 1.1.** *Assume that the Hamiltonian vector field  $\vec{H}$  is monotone and its flow  $\varphi_t$  does not contain any conjugate point on  $M$ . Then there are (measurable) Lagrangian distributions  $\Delta^\pm$  of  $M$  which are invariant under  $d\varphi_t$ .*

By the work of [5], we can define the reduced curvature  $\tilde{\mathfrak{R}}$  of a monotone Hamiltonian vector field  $\vec{H}$  (see Section 6 for the definition). Under the assumptions of Theorem 1.1, we can show that the integral

of the trace  $\tilde{\mathfrak{r}}$  of  $\tilde{\mathfrak{R}}$  with respect to any invariant measure (in particular the Liouville measure) of  $\vec{H}$  is non-positive. Moreover, this integral vanishes only if  $\tilde{\mathfrak{r}}$  vanishes. This extends the results of [16, 14, 13] to our setting. More precisely, we have

**Theorem 1.2.** *Let  $c$  be a regular value of  $H$ . Let  $\mu$  be an invariant measure of the flow  $\varphi_t$  of  $\vec{H}$  on  $\Sigma_c := H^{-1}(c)$ . Assume that  $\varphi_t$  has no conjugate point on the support of  $\mu$ . Then the following holds*

$$\int_{\Sigma_c} \mathfrak{r}_\alpha d\mu(\alpha) \leq 0.$$

Moreover, equality holds only if  $\mathfrak{r} \equiv 0$  on the support of  $\mu$ .

We also show that the flow of the Hamiltonian vector field  $\vec{H}$  is Anosov assuming that the reduced curvature is negative.

**Theorem 1.3.** *Let  $c$  be a regular value of  $H$ . Assume that the Hamiltonian vector field is monotone and the reduced curvature is bounded above and below by two negative constants on  $\Sigma_c$ . Then the flow of  $\vec{H}$  is Anosov on  $\Sigma_c$ .*

The above theorem is proved in [3] under the assumption that  $\Sigma_c$  is compact. We give a different proof which relax this compactness assumption to a lower curvature bound.

If the invariant distributions  $\Delta^\pm$  defined in Theorem 1.1 are everywhere transversal, then it was shown in [10] that the geodesic flow is Anosov. An extension of this result can also be found in [9]. By combining a reduction procedure together with the analysis in [10], we obtain the following result.

**Theorem 1.4.** *Suppose that the assumption of Theorem 1.1 are satisfied. Let  $c$  be a regular value of  $H$  and assume that  $\Sigma_c = H^{-1}(c)$  is compact. Then the followings are equivalent.*

- (1) *The flow  $\varphi_t$  is Anosov on  $\Sigma_c$ ,*
- (2)  *$\Delta^+$  and  $\Delta^-$  are transversal in  $T\Sigma_c$ ,*
- (3)  *$\Delta^+ \cap \Delta^- = \text{span}\{\vec{H}\}$ .*

Under the assumption that the reduced curvature is everywhere non-positive, we also obtain the following which generalize another result of [10].

**Theorem 1.5.** *Assume that the monotone Hamiltonian vector field  $\vec{H}$  has non-positive reduced curvature. Then the flow  $\varphi_t$  of  $\vec{H}$  is Anosov if and only if, for each  $\alpha$  in  $M$ , there is a time  $t$  such that the reduced*

curvature  $\tilde{\mathfrak{R}}$  of  $\vec{H}$  satisfies  $\langle \tilde{\mathfrak{R}}_{\varphi_t(\alpha)} \tilde{v}, \tilde{v} \rangle < 0$  for some  $t$  and for some vector  $\tilde{v}$  in  $\tilde{\Lambda}_{\varphi_t(\alpha)}$ .

Using the result in Theorem 1.4, we can estimate the measure theoretic entropy for invariant measures of  $\varphi_t$  in terms of the reduced curvature  $\tilde{\mathfrak{R}}$ . This generalizes the corresponding results in [11] and [13].

**Theorem 1.6.** *Let  $c$  be a regular value of  $H$ . Let  $\mu$  be an invariant measure of the flow  $\varphi_t$  of  $\vec{H}$  on the compact manifold  $\Sigma_c := H^{-1}(c)$ . Assume that  $\varphi_t$  has no conjugate point on the support of  $\mu$ . Then the following holds*

$$h_\mu(\varphi_t) \leq (n-1)^{1/2} \left( - \int_{\Sigma_c} \text{tr} \tilde{\mathfrak{R}}_\alpha d\mu(\alpha) \right)^{1/2}.$$

Moreover, equality holds only if  $\tilde{\mathfrak{R}}$  is constant on the support of  $\mu$ .

We remark that a lower estimate under the assumption that the reduced curvature is non-positive was done in [8] which generalizes the earlier work of [7, 18].

Finally, we also show that the following generalization of the result in [19] is also possible.

**Theorem 1.7.** *Let  $c$  be a regular value of  $H$ . Let  $\mu$  be an invariant measure of the flow  $\varphi_t$  of  $\vec{H}$  on the compact manifold  $\Sigma_c := H^{-1}(c)$ . Then the following holds*

$$h_\mu(\varphi_t) \leq \frac{1}{2} \int_{\Sigma_c} \sum_{i=1}^{n-1} |1 - \lambda_i(\alpha)| d\mu(\alpha)$$

for any invariant measure  $\mu$  of  $\varphi_t$  on  $\Sigma_c$  and where  $\lambda_i(\alpha)$  are eigenvalues of the operator  $\tilde{\mathfrak{R}}_\alpha$ .

The content of this paper is as follows. In Section 3, we discuss some materials on curves in Lagrangian Grassmannian which are needed in the definition of the curvature of  $\vec{H}$ . In section 4, we recall several basic results on linear second order ODEs which are needed in this paper. In Section 5, we recall the definition of the curvature of  $\vec{H}$ . In Section 6, we recall a reduction procedure studied in [4] which is needed for the proof of the above theorems. Finally, sections 7-13 are devoted to the proofs.

## 2. NOTATIONS

$\mathcal{V}$  a symplectic vector space  
 $M$  a symplectic manifold  
 $\omega$  symplectic form on  $\mathcal{V}$  or on  $M$   
 $\mathcal{L}(\mathcal{V})$  Lagrangian Grassmannian of  $\mathcal{V}$   
 $J$  curve in  $\mathcal{L}(\mathcal{V})$   
 $J^\circ$  derivative curve of  $J$   
 $R$  curvature operator of  $J$   
 $\mathcal{R}$  matrix representation of  $R$   
 $\langle \cdot, \cdot \rangle^t$  the canonical bilinear form on  $J(t)$   
 $e^1(t), \dots, e^n(t)$  a canonical frame of a regular curve  $J$   
 $f^i(t) = \dot{e}^i(t)$   
 $H$  Hamiltonian  
 $\vec{H}$  Hamiltonian vector field of  $H$   
 $J_\alpha$  Jacobi curve of  $\vec{H}$  at  $\alpha$   
 $R_\alpha(t)$  curvature operator of  $J_\alpha$   
 $\mathfrak{R}$  curvature operator of  $\vec{H}$   
 $\tilde{J}_\alpha$  reduced Jacobi curve of  $\vec{H}$  at  $\alpha$   
 $\tilde{R}_\alpha(t)$  curvature operator of  $\tilde{J}_\alpha$   
 $\mathfrak{R}$  curvature operator of  $\vec{H}$

## 3. REGULAR CURVES IN LAGRANGIAN GRASSMANNIAN

Let  $\mathcal{V}$  be a  $2n$ -dimensional vector space equipped with a symplectic form  $\omega$ . The space of all Lagrangian subspaces in  $\mathcal{V}$ , called Lagrangian Grassmannian, is denoted by  $\mathcal{L} = \mathcal{L}(\mathcal{V})$ . In this appendix, we recall the definition and properties of regular curves in  $\mathcal{L}$ . For a more complete discussion, see [3, 17].

A smooth curve  $t \mapsto J(t)$  in  $\mathcal{L}$  carries a family of canonical bilinear forms  $\langle \cdot, \cdot \rangle^t$  defined by

$$(3.1) \quad \langle v_1, v_2 \rangle^t := \omega(\dot{v}_1(t), v_2)$$

for all  $v_1$  and  $v_2$  in  $J(t)$ , where  $t \mapsto v_1(t)$  is a curve satisfying  $v_1 = v_1(t)$ .

**Definition 3.1.** A smooth curve  $t \mapsto J(t)$  in the Lagrangian Grassmannian  $\mathcal{L}$  is *regular* if the bilinear form (3.1) is non-degenerate for each  $t$ .

Recall that a basis  $e_1, \dots, e_n, f_1, \dots, f_n$  in  $\mathcal{V}$  is a symplectic basis if  $\omega(e_i, e_j) = \omega(f_i, f_j) = 0$  and  $\omega(f_i, e_j) = \delta_{ij}$ . For a regular curve  $J$ , one can also define a canonical frame in  $J$  which is unique up to transformations by orthogonal matrices. In fact, canonical frames can be found for more general curves, see [17] for more detail.

**Proposition 3.2.** *Assume that  $J$  is a regular curve in the Lagrangian Grassmannian  $\mathcal{L}$ . Then there exists a smooth family of basis*

$$E(t) = (e^1(t), \dots, e^n(t))^T \quad (T \text{ denotes transpose})$$

*on  $J(t)$  orthonormal with respect to the inner product  $\langle \cdot, \cdot \rangle^t$  such that*

$$\{e^1(t), \dots, e^n(t), \dot{e}^1(t) := f^1(t), \dots, \dot{e}^n(t) := f^n(t)\}$$

*forms a Darboux basis for the symplectic vector space  $\mathcal{V}$  such that  $\ddot{e}^i(t)$  is contained in  $J(t)$  for  $i = 1, \dots, n$ . Moreover, if  $\bar{E}(t) = (\bar{e}^1(t), \dots, \bar{e}^n(t))^T$  is another such family, then there exists an orthogonal matrix  $U$  such that  $\bar{E}(t) = U E(t)$ .*

*Proof.* Let us fix a family of basis  $\tilde{E}(t) = (\tilde{e}^1(t), \dots, \tilde{e}^n(t))^T$  on  $J_\alpha(t)$  orthonormal with respect to the canonical inner product. Since  $J(t)$  is a Lagrangian subspace, we have

$$(3.2) \quad \omega(\tilde{e}^i(t), \tilde{e}^j(t)) = 0.$$

Since  $\tilde{E}(t)$  is orthonormal with respect to the inner product  $\langle \cdot, \cdot \rangle^t$ , we also have

$$(3.3) \quad \omega(\dot{\tilde{e}}^i(t), \tilde{e}^j(t)) = \delta_{ij}.$$

Let  $U(t)$  be any smooth family of orthogonal matrices, let  $E(t) = U(t)\tilde{E}(t)$ , and let  $f^i(t) = \dot{e}^i(t)$ . Let  $\Omega(t)$  be the matrix with  $ij$ -th entry equal to  $\omega(\dot{e}^i(t), \dot{e}^j(t))$ . Then, by (3.2) and (3.3), we have

$$\Omega(t) = -\dot{U}(t)U(t)^T + U(t)\dot{U}(t)^T + U(t)\tilde{\Omega}(t)U(t)^T.$$

Therefore, if we let  $U(t)$  be the solution of

$$(3.4) \quad \dot{U}(t) = \frac{1}{2}U(t)\tilde{\Omega}(t)$$

with  $U(0)$  orthogonal. Then  $\Omega(t) \equiv 0$  (Note that  $\tilde{\Omega}(t)$  is skew-symmetric. Therefore,  $U(t)$  is orthogonal). Hence

$$e^1(t), \dots, e^n(t), f^1(t), \dots, f^n(t)$$

is a Darboux basis.

Finally, if we assume that  $\tilde{e}_1(t), \dots, \tilde{e}_n(t), \dot{\tilde{e}}_1(t), \dots, \dot{\tilde{e}}_n(t)$  is a Darboux basis of  $\mathcal{V}$  for each  $t$ . Then  $\dot{U}(t) = 0$  and the uniqueness claim follows.  $\square$

Proposition 3.2 leads to the following definition.

**Definition 3.3.** A family of basis  $e^1(t), \dots, e^n(t)$  of a regular Jacobi curve  $J(t)$  is a *canonical frame* of  $J$  if

- (1)  $\{e^1(t), \dots, e^n(t), f^1(t) := \dot{e}^1(t), \dots, f^n(t) := \dot{e}^n(t)\}$  forms a Darboux basis for the symplectic vector space  $\mathcal{V}$ ,

(2)  $\ddot{e}^i(t)$  is contained in  $J(t)$  for  $i = 1, \dots, n$ .

The proof of Proposition 3.2 gives the following result which will be needed later.

**Lemma 3.4.** *Let  $\tilde{E}(t) := (\tilde{e}^1(t), \dots, \tilde{e}^n(t))^T$  be a family of orthonormal basis (with respect to  $\langle \cdot, \cdot \rangle^t$ ) in a curve  $J(\cdot)$  of the Lagrangian Grassmannian  $\mathcal{L}$ . Let  $\Omega$  be the matrix with  $ij$ -th entry equal to  $\omega(\tilde{e}^i(t), \dot{\tilde{e}}^j(t))$ . Let  $U$  be a solution of*

$$(3.5) \quad \dot{U}(t) = \frac{1}{2}U(t)\Omega(t).$$

*Then  $E(t) = U(t)\tilde{E}(t)$  forms a canonical frame.*

Proposition 3.2 also allows us to make the following definitions.

**Definition 3.5.** Let  $e^1(t), \dots, e^n(t)$  be a canonical frame of a regular curve  $J$ . The curve  $J^o(\cdot)$  in  $\mathcal{L}(\mathcal{V})$  defined by

$$J^o(t) := \text{span}\{f^1(t), \dots, f^n(t)\}$$

is called the *derivative curve* of  $J$ .

The canonical frame satisfies a second order equation.

**Proposition 3.6.** *Let  $E(t) = (e^1(t), \dots, e^n(t))^T$  be a canonical frame. Then there is a linear operator  $R(t) : J(t) \rightarrow J(t)$  symmetric with respect to the inner product  $\langle \cdot, \cdot \rangle^t$  such that*

$$\dot{e}^i(t) = f^i(t), \quad \dot{f}^i(t) = -R(t)e^i(t).$$

*Proof.* By the definition of canonical frame,  $\dot{f}^i(t)$  is contained in  $J(t)$ . Therefore, we can define  $R(t)$  by

$$R(t)e^i(t) = -\dot{f}^i(t).$$

By Theorem 3.2, this definition of  $R(t)$  is independent of the choice of canonical Darboux frames.

Finally, using the equation  $\omega(f^i(t), f^j(t)) = 0$  and differentiating with respect to time, we see that

$$\omega(\dot{e}^j(t), R(t)e^i(t)) = \omega(\dot{e}^i(t), R(t)e^j(t)).$$

It follows that  $R(t)$  is symmetric with respect to the canonical inner product.  $\square$

**Definition 3.7.** The equations

$$\dot{e}^i(t) = f^i(t), \quad \dot{f}^i(t) + R(t)e^i(t) = 0$$

in Proposition 3.6 are called *structural equations* of the curve  $J$ . The operators  $R(t)$  are the *curvature operators* of  $J$ . The matrix representation of  $R(t)$  is denoted by  $\mathcal{R}(t)$  and it is defined by

$$R(t)e^i(t) = \sum_{j=1}^n \mathcal{R}^{ij}(t)e^j(t)$$

#### 4. ON SECOND ORDER EQUATIONS

Let  $E(t)$  be a canonical frame of a regular curve  $J$ . Then any vector  $\xi$  in the symplectic vector space  $\mathcal{V}$  can be written as

$$(4.1) \quad \xi = -\dot{a}(t)^T E(t) + a(t)^T F(t)$$

for a family of vectors  $a(t)$ . Moreover,  $a(t)$  satisfies

$$(4.2) \quad \ddot{a}(t) = -\mathcal{R}(t)a(t).$$

In this section, we recall some facts on the fundamental solutions of the equation (4.2). More precisely, let  $B$  be the matrix solution of the equation

$$(4.3) \quad \ddot{B}(t) + \mathcal{R}(t)B(t) = 0$$

with initial conditions  $B(0) = 0$  and  $\dot{B}(0) = I$ .

For  $t \neq 0$ , let  $S(t) := \dot{B}(t)B(t)^{-1}$ . Then  $S(t)$  is a family of symmetric matrices satisfying

$$(4.4) \quad \dot{S}(t) + S(t)^2 + \mathcal{R}(t) = 0.$$

Let  $D(s, t)$  be defined by

$$(4.5) \quad D(s, t) = B(t) \int_t^s B(\tau)^{-1} (B(\tau)^{-1})^T d\tau.$$

From now on, we denote the derivative with respect to  $t$  and  $s$  by dot and prime, respectively. For instance,  $\dot{D}$  denotes derivative of  $D$  with respect to  $t$  and  $D'$  denotes derivative with respect to  $s$ .

**Lemma 4.1.** *The family of matrices  $t \mapsto D(s, t)$  is a solution of the equation (4.3) which satisfies the boundary conditions*

$$D(s, 0) = I, \quad D(s, s) = 0, \quad \dot{D}(s, s) = -(B(s)^{-1})^T.$$

*Proof.* A computation shows that  $t \mapsto D(s, t)$  is a solution of the equation (4.3) which satisfies the conditions  $D(s, s) = 0$  and  $\dot{D}(s, s) = -(B(s)^{-1})^T$ . Since the Wronskian  $\dot{D}(s, t)^T B(t) - D(s, t)^T \dot{B}(t)$  is independent of time  $t$ , we also have  $D(s, 0) = I$ .  $\square$



Let  $U(s, t) = \dot{D}(s, t)D(s, t)^{-1}$ .  $U(s, t)$  is the solution of the equation

$$(4.6) \quad \dot{U}(s, t) + U(s, t)^2 + \mathcal{R}(t) = 0.$$

Next, we apply the following comparison principle of matrix Riccati equations (see [21] for the proof).

**Theorem 4.2.** *Let  $A_i(t)$  be a family square matrices. Let  $S_i$  be the solution of the matrix Riccati equation*

$$\dot{S}_i(t) + A_i(t)S_i(t)^2 + \mathcal{R}_i(t) = 0, \quad i = 1, 2.$$

*Assume that  $S_2(t_0) \geq S_1(t_0)$  (resp.  $S_1(t_0) \geq S_2(t_0)$ ) for some  $t_0$  and  $\mathcal{R}_1(t) \geq \mathcal{R}_2(t)$ ,  $A_1(t) \geq A_2(t)$  for all  $t \geq t_0$ . Then*

$$S_2(t) \geq S_1(t) \quad (\text{resp. } S_1(t) \geq S_2(t))$$

*for all  $t \geq t_0$  (resp.  $t \leq t_0$ ).*

For the rest of this section, we assume that any solution  $B(\cdot)$  of the equation (4.3) satisfies the following assumption.

*Assumption 4.3.* If  $B(t_0) = 0$  and  $\det \dot{B}(t_0) \neq 0$  for some  $t_0$ , then  $\det B(t) \neq 0$  for all  $t \neq t_0$ .

Under this assumption, the matrix  $U(s, t)$  is invertible whenever  $s \neq t$ .

**Lemma 4.4.** *Assume that  $s_1 < s_2 < 0 < s_3 < s_4$ . Then, under Assumption 4.3,*

$$U(s_2, t) \geq U(s_1, t) \geq U(s_4, t) \geq U(s_3, t)$$

*for all  $t$  in the open interval  $(s_2, s_3)$ .*

*Proof.* By the matrix Riccati equation (4.6),  $\dot{U}(s, t) < 0$  for all  $t$  near  $s$ . It follows that the eigenvalues of  $U(s, t)$  goes to  $+\infty$  as  $t \rightarrow s^+$  and goes to  $-\infty$  as  $t \rightarrow s^-$ . The result follows from Theorem 4.2.  $\square$

It follows from the above lemma that we can define the following

$$U^+(t) := \lim_{s \rightarrow +\infty} U(s, t), \quad U^-(t) := \lim_{s \rightarrow -\infty} U(s, t).$$

The comparison theorem also gives the following estimate.

**Lemma 4.5.** *Assume that  $\mathcal{R}(t) \geq -k^2 I$  for some constant  $k > 0$ . Then*

$$\begin{aligned} k \coth(kt) I &\geq S(t) > U^-(t) \geq U^+(t) \\ (\text{resp. } -k \coth(kt) I &\leq S(t) < U^+(t) \leq U^-(t)) \end{aligned}$$

*for all  $t > 0$  (resp.  $t < 0$ ).*

*Proof.* The family  $t \mapsto k \coth(kt)I$  is a solution of (4.3) with  $R(t) = -k^2I$ . The rest follows from Theorem 4.2.  $\square$

Let  $D^\pm$  be the solutions of the equation

$$\dot{D}^\pm(t) = U^\pm(t)D^\pm(t)$$

with initial condition  $D^\pm(0) = I$ .

**Lemma 4.6.** *Assume that there are non-negative constants  $k$  and  $K$  such that  $-K^2I \geq \mathcal{R}(t) \geq -k^2I$ . Then  $U^+$  (resp.  $U^-$ ) satisfies the following*

$$-KI \geq U^+(t) \geq -kI \quad (\text{resp. } kI \geq U^-(t) \geq KI)$$

for all  $t$  and

$$|b|e^{-kt} \leq |D^+(t)b| \leq |b|e^{-Kt} \quad (\text{resp. } |b|e^{Kt} \leq |D^-(t)b| \leq |b|e^{kt})$$

for any vector  $b$  and all  $t > 0$ .

*Proof.* We will only prove the case when  $K > 0$  since the case  $K = 0$  is very similar. Note that  $K \coth(K(t-s))I$  is a solution of (4.3) with  $\mathcal{R} = -K^2I$ . Therefore, by Theorem 4.2, we have

$$K \coth(K(t-s))I \geq U(s, t) \geq k \coth(k(t-s))I$$

for all  $t < s$ .

Therefore, if we let  $s \rightarrow \infty$ , then we obtain

$$(4.7) \quad -KI \geq U^+(t) \geq -kI.$$

It follows from (4.7) that the Euclidean norm  $|D^+(t)b|$  of  $D^+(t)b$  satisfies

$$\begin{aligned} \frac{d}{dt}|D^+(t)b|^2 &= 2 \langle U^+(t)D^+(t)b, D^+(t)b \rangle \\ &\leq -2K|D^+(t)b|^2. \end{aligned}$$

Therefore, it follows that

$$|D^+(t)b|^2 \leq |b|^2 e^{-2Kt}.$$

$\square$

Finally, we show that  $|B(t)v|$  goes to  $+\infty$  uniformly as  $t$  goes to  $\pm\infty$ .

**Lemma 4.7.** *Let  $B(\cdot)$  be a solution of (4.3) with  $B(0) = 0$ ,  $\dot{B}(0) = I$ , and  $R(t) \geq -k^2I$ . Then, for each number  $K > 0$ , there is  $T > 0$  such that*

$$|B(t)v| \geq K|v|$$

for all  $t \geq T$  (resp.  $t \leq -T$ ).

*Proof.* Let  $D^+$  be the solution of (4.3) with initial condition  $D^+(0) = I$  and  $\dot{D}^+(0) = U^+(0)$ . It follows from the definition of  $D(s, t)$  that  $\lim_{s \rightarrow +\infty} D(s, t) = D^+(t)$  and

$$D^+(t) = B(t) \int_t^\infty B(\tau)^{-1} (B(\tau)^{-1})^T d\tau.$$

If we differentiate this equation with respect to time  $t$ , then we obtain

$$U^+(t) D^+(t) = -(B(t)^{-1})^T + S(t) D^+(t).$$

Therefore, the following holds

$$S(t) - U^+(t) = (B(t)^{-1})^T M(t)^{-1} B(t)^{-1},$$

where  $M(t) = \int_t^\infty B(\tau)^{-1} (B(\tau)^{-1})^T d\tau$ .

It follows from Lemma 4.5 that there is  $t_0 > 0$  such that

$$\begin{aligned} 4k &\geq |\langle S(t)v - U^+(t)v, v \rangle| \\ &= |\langle M(t)^{-1} B(t)^{-1} v, B(t)^{-1} v \rangle| \\ &\geq \frac{|B(t)^{-1} v|^2}{\|M(t)\|} \end{aligned}$$

for all  $t > t_0$ . Here  $\|M(t)\|$  denotes the operator norm of  $M(t)$ .

Therefore, for all  $v$  satisfying  $|v| = 1$ , we have

$$|B(t)v| \geq \frac{1}{\|B(t)^{-1}\|} \geq \frac{1}{(4k\|M(t)\|)^{1/2}} \rightarrow \infty$$

as  $t \rightarrow \infty$ . □

## 5. MONOTONE HAMILTONIAN VECTOR FIELDS

Let  $M$  be a symplectic manifold equipped with a symplectic structure  $\omega$  and a Lagrangian distribution  $\Lambda$ . Let  $H : M \rightarrow \mathbb{R}$  be Hamiltonian and let  $\vec{H}$  be the corresponding Hamiltonian vector field defined by

$$\omega(\vec{H}, \cdot) = -dH(\cdot).$$

Let us consider the canonical bilinear form  $\langle \cdot, \cdot \rangle$  of the Hamiltonian vector field  $\vec{H}$  defined on  $\Lambda$  by

$$(5.1) \quad \langle v_1, v_2 \rangle_\alpha = \omega_\alpha([\vec{H}, V_1], V_2),$$

where  $V_1$  and  $V_2$  are two sections of  $\Lambda$  such that  $V_1(\alpha) = v_1$  and  $V_2(\alpha) = v_2$ . Since  $\Lambda$  is a Lagrangian distribution and the Hamiltonian vector field  $\vec{H}$  preserves  $\omega$ , the above bilinear form is well-defined.

**Definition 5.1.** We say that the Hamiltonian vector field  $\vec{H}$  is *monotone* if the above bilinear form is a Riemannian metric on  $\Lambda$ .

In this section, following the approach introduced by [5], we consider the curvature of monotone Hamiltonian vector fields. For this, let  $\varphi_t$  be the flow of the Hamiltonian vector field  $\vec{H}$ , let us fix a point  $\alpha$  in the manifold  $M$  and consider the following curve of Lagrangian subspaces in the Lagrangian Grassmannian  $\mathcal{L}(T_\alpha M)$ .

**Definition 5.2.** The curve  $t \mapsto J_\alpha(t)$  in the Lagrangian Grassmannian  $\mathcal{L}(T_\alpha M)$  defined by

$$J_\alpha(t) := d\varphi_t^{-1}(\Lambda_{\varphi_t(\alpha)})$$

is called the *Jacobi curve* of  $\vec{H}$  at  $\alpha$ .

**Proposition 5.3.** *The canonical bilinear form (5.1) of the Hamiltonian vector field  $\vec{H}$  and the canonical bilinear form of the Jacobi curve defined by (3.1) are related by*

$$\langle v_1, v_2 \rangle_{\varphi_t(\alpha)} = \langle d\varphi_t^{-1}v_1, d\varphi_t^{-1}v_2 \rangle^t$$

for all  $v_1$  and  $v_2$  in  $T_{\varphi_t(\alpha)}M$ .

*In particular, if the canonical bilinear form (5.1) is everywhere non-degenerate, then the Jacobi curve  $J_\alpha(t)$  is regular for each  $\alpha$ .*

*Proof.* Let  $e^1(t), \dots, e^n(t)$  be given by Proposition 3.2. Let  $V_t^i$  be a time dependent vector field on  $M$  such that  $d\varphi_t(e^i(t)) = V_t^i(\varphi_t(\alpha))$ . It follows from the definition of the bilinear form (3.1) and the invariance of the form  $\omega$  under the flow  $\varphi_t$  that

$$\begin{aligned} \langle \varphi_t^* V_t^i, \varphi_t^* V_t^i \rangle^t &= \omega_\alpha(\varphi_t^*([\vec{H}, V_t^i] + \dot{V}_t^i), \varphi_t^* V_t^i) \\ &= \omega_{\varphi_t(\alpha)}([\vec{H}, V_t^i], V_t^i) \\ &= \langle V_t^i, V_t^i \rangle_{\varphi_t(\alpha)}. \end{aligned}$$

□

It is, therefore, natural to call a Hamiltonian vector field  $\vec{H}$  regular if the corresponding canonical bilinear form is non-degenerate. In particular, if  $\vec{H}$  is monotone, then it is regular.

**Definition 5.4.** Assuming that the Hamiltonian vector field  $\vec{H}$  is regular. Let us denote by  $J_\alpha^o(t)$  the derivative curve of the Jacobi curve  $J_\alpha(t)$  at  $\alpha$ . We define a Lagrangian distribution  $\Lambda^o$  by

$$\Lambda^o = J_\alpha^o(0).$$

We will refer to distributions  $\Lambda$  and  $\Lambda^o$  as the *vertical* and the *horizontal* bundles, respectively. We will also refer a tangent vector in the distributions  $\Lambda$  and  $\Lambda^o$  a vertical vector and a horizontal vector, respectively. If  $w$  is a tangent vector in  $TM = \Lambda \oplus \Lambda^o$ , then its components

$w^v$  in  $\Lambda$  and  $w^h$  in  $\Lambda^o$  are called vertical and horizontal parts of  $w$ , respectively.

The Jacobi curve  $J_\alpha$  and the derivative curve  $J_\alpha^o$  satisfy the following property.

**Proposition 5.5.** *For each  $\alpha$  in  $M$ , we have*

$$d\varphi_s(J_\alpha(t)) = J_{\varphi_s(\alpha)}(t-s), \quad d\varphi_s(J_\alpha^o(t)) = J_{\varphi_s(\alpha)}^o(t-s).$$

*Proof.* Let  $J_\alpha(t)$  be the Jacobi curve at  $\alpha$ . It follows that

$$d\varphi_s(J_\alpha(t)) = d\varphi_s(d\varphi_t^{-1}\Lambda_{\varphi_t(\alpha)}) = d\varphi_{t-s}^{-1}(\Lambda_{\varphi_{t-s}(\varphi_s(\alpha))}) = J_{\varphi_s(\alpha)}(t-s).$$

It also follows that  $d\varphi_s(e^1(s+t)), \dots, d\varphi_s(e^n(s+t))$  is a canonical frame of  $J_{\varphi_s(\alpha)}$ . The second assertion follows from this.  $\square$

Similarly, we define the curvature operator of a Hamiltonian vector field by that of the Jacobi curves.

**Definition 5.6.** Assuming that the Hamiltonian vector field  $\vec{H}$  is regular. Let  $R_\alpha(t)$  be the curvature operators of the Jacobi curve  $J_\alpha(t)$  at  $\alpha$ . The *curvature operator*  $\mathfrak{R} : \Lambda \rightarrow \Lambda$  of  $\vec{H}$  is defined by

$$\mathfrak{R}_\alpha = R_\alpha(0).$$

**Proposition 5.7.** *Assuming that the Hamiltonian vector field  $\vec{H}$  is regular. For each  $\alpha$  in  $M$  and each vector  $v$  in  $T_{\varphi_t(\alpha)}M$ , the following holds.*

$$R_\alpha(t)(d\varphi_t(v)) = d\varphi_t^{-1}(\mathfrak{R}_{\varphi_t(\alpha)}(v)).$$

Moreover, for each vertical vector field  $V$ , the curvature operator  $\mathfrak{R}$  satisfies

$$\mathfrak{R}_\alpha(V) = -[\vec{H}, [\vec{H}, V]^h]^v(\alpha).$$

*Proof.* Let  $e^1(t), \dots, e^n(t), f^1(t), \dots, f^n(t)$  be given by Proposition 3.2. By the proof of Proposition 5.5,  $t \mapsto (d\varphi_s(e^1(t+s)), \dots, d\varphi_s(e^n(t+s)))$  is a canonical Darboux frame at  $\varphi_t(\alpha)$ . Therefore, we have

$$\begin{aligned} d\varphi_s(R_\alpha(t+s)e^i(t+s)) &= -\frac{d^2}{dt^2}d\varphi_s(e^i(t+s)) \\ &= R_{\varphi_s(\alpha)}(t)d\varphi_s(e^i(t+s)). \end{aligned}$$

If we set  $t = 0$ , then we obtain

$$d\varphi_s(R_\alpha(s)e^i(s)) = R_{\varphi_s(\alpha)}(0)d\varphi_s(e^i(s))$$

and the first assertion follows.

Since the Hamiltonian vector field  $\vec{H}$  is regular, it is transversal to  $\Lambda$ . It follows that there is a vector field  $V^i$  on  $M$  such that  $d\varphi_t(e^i(t)) =$

$V^i(\varphi_t(\alpha))$ . It follows from the definition of the canonical Darboux frame that

$$f^i(t) = \dot{e}^i(t) = \varphi_t^*([\vec{H}, V^i])(\alpha)$$

and

$$R_\alpha(t) e^i(t) = -\varphi_t^*([\vec{H}, [\vec{H}, V^i]])(\alpha).$$

Therefore,

$$\mathfrak{R}_\alpha(V^i(\alpha)) = -[\vec{H}, [\vec{H}, V^i]](\alpha) = -[\vec{H}, [\vec{H}, V^i]^h]^v(\alpha).$$

It remains to note that the maps  $v \mapsto [\vec{H}, V]^h(\alpha)$  and  $w \mapsto [\vec{H}, W]^v(\alpha)$  are tensorial on  $\Lambda$  and  $\Lambda^o$ , respectively.  $\square$

## 6. REDUCTION OF CURVES IN LAGRANGIAN GRASSMANNIAN

Let  $v$  be a vector in a symplectic vector space  $\mathcal{V}$ . Let  $v^\perp$  be the symplectic complement of  $v$ . Recall that the symplectic reduction  $\tilde{\mathcal{V}}$  of  $\mathcal{V}$  by  $v$  is defined by

$$\tilde{\mathcal{V}} = (\mathcal{V} \cap v^\perp) / \mathbb{R} v.$$

The symplectic form  $\omega$  descends to a symplectic form  $\tilde{\omega}$  on  $\tilde{\mathcal{V}}$ . It follows that any Lagrangian subspace in  $\mathcal{V}$  also descends to a Lagrangian subspace in  $\tilde{\mathcal{V}}$ . In particular, if  $J$  is a curve in the Lagrangian Grassmannian  $\mathcal{L}(\mathcal{V})$ , then it descends to a curve  $\tilde{J}$  in  $\mathcal{L}(\tilde{\mathcal{V}})$ . Note also that the canonical bilinear form (3.1) of the curve  $J$  clearly descends to that of the curve  $\tilde{J}$ . It follows that  $\tilde{J}$  is regular if  $J$  is. Therefore, there is a curvature operator for the curve  $\tilde{J}$  which is denoted by  $\tilde{R}$ . For the rest of this section, we recall how the curvature of  $J$  relates to that of  $\tilde{J}$ . The reduced Jacobi curve was considered in [4]. Here we give slightly different proofs of the results.

By Proposition 3.6, we can find a canonical frame

$$(6.1) \quad \tilde{e}^1(t), \dots, \tilde{e}^{n-1}(t)$$

and a curvature operator  $\tilde{R}(t) : \tilde{J}(t) \rightarrow \tilde{J}(t)$  satisfying

$$\dot{\tilde{e}}^i(t) = \tilde{f}^i(t), \quad \dot{\tilde{f}}^i(t) = -\tilde{R}(t)\tilde{e}^i(t).$$

Assume that  $v$  is transversal to the  $J(t)$  for all  $t$ . It follows there is a family of bases along  $J(t)$ , denoted by

$$\tilde{e}^1(t), \dots, \tilde{e}^n(t),$$

which is orthonormal with respect to the canonical bilinear form (3.1) such that the first  $n-1$  of them descend to the canonical frame (6.1) of  $\tilde{J}(t)$ . Let  $\Omega(t)$  be the matrix with  $ij$ -th entry defined by  $\Omega_{ij}(t) := \omega(\tilde{e}^i(t), \dot{\tilde{e}}^j(t))$  and let  $U$  be the solution of the equation (3.5) with initial

condition  $U(0) = I$ . Note that  $\Omega_{ij}(t) = 0$  unless  $i \neq n$  and  $j \neq n$ . Therefore, we can let  $\bar{\Omega}(t)$  be the  $n-1$ -vector with  $i$ -th entry defined by  $\bar{\Omega}_i(t) := \Omega_{ni}(t)$ . The curvature of the curve  $J$  and its reduction  $\tilde{J}$  are related as follows.

**Proposition 6.1.**

$$U(t)^T \mathcal{R}(t) U(t) = \begin{pmatrix} \tilde{\mathcal{R}}(t) - \frac{3}{4} \bar{\Omega}(t) \otimes \bar{\Omega}(t) & \frac{1}{2} \dot{\bar{\Omega}}(t) \\ \frac{1}{2} \bar{\Omega}(t)^T & \frac{1}{4} |\bar{\Omega}(t)|^2 \end{pmatrix}$$

Here  $\mathcal{R}(t)$  and  $\tilde{\mathcal{R}}(t)$  denote the matrix representations of  $R(t)$  and  $\tilde{R}(t)$ , respectively.  $\bar{\Omega} \otimes \bar{\Omega}$  is the matrix defined by

$$\bar{\Omega} \otimes \bar{\Omega}(w) = \langle \bar{\Omega}, w \rangle \bar{\Omega}.$$

*Proof.* Let  $\tilde{E}(t) = (\tilde{e}^1(t), \dots, \tilde{e}^n(t))^T$  and let  $\tilde{F}(t) = (\tilde{f}^1(t), \dots, \tilde{f}^n(t))^T$ . By assumption  $\omega(v, \tilde{e}^i(t)) = 0$  and it follows that  $\omega(v, \dot{\tilde{e}}^i(t)) = 0$  for all  $i \neq n$ . Therefore, we have

$$(6.2) \quad v = b(t) \left( \dot{\tilde{e}}^n(t) + \sum_{j=1}^{n-1} \Omega_{nj}(t) \tilde{e}^j(t) \right)$$

where  $b(t)$  is the length of the  $J(t)$ -component of  $v$ .

If we differentiate the above equation with respect to  $t$ , then we obtain

$$\ddot{\tilde{e}}^n(t) = - \sum_{j=1}^{n-1} \left( \frac{\dot{b}(t)}{b(t)} \Omega_{nj}(t) + \dot{\Omega}_{nj}(t) \right) \tilde{e}^j(t) - \sum_{j=1}^{n-1} \Omega_{nj}(t) \dot{\tilde{e}}^j(t) - \frac{\dot{b}(t)}{b(t)} \dot{\tilde{e}}^n(t).$$

Since  $\langle \tilde{e}^n(t), \tilde{e}^n(t) \rangle^t = 1$ , it follows from the above that  $\dot{b} = 0$ . Therefore,

$$(6.3) \quad \ddot{\tilde{e}}^n(t) = - \sum_{j=1}^{n-1} \dot{\Omega}_{nj}(t) \tilde{e}^j(t) - \sum_{j=1}^{n-1} \Omega_{nj}(t) \dot{\tilde{e}}^j(t).$$

On the other hand, by the definition of  $\tilde{e}^i(t)$  and (6.2), we have

$$\begin{aligned} \ddot{\tilde{e}}^i(t) &= - \sum_{j=1}^{n-1} \tilde{\mathcal{R}}_{ij}(t) \tilde{e}^j(t) + a_i(t) v \\ &= - \sum_{j=1}^{n-1} \left( \tilde{\mathcal{R}}_{ij}(t) - a_i(t) b(t) \Omega_{nj}(t) \right) \tilde{e}^j(t) + a_i(t) b(t) \dot{\tilde{e}}^n(t). \end{aligned}$$

for some functions  $a_i$  and for all  $i \neq n$ .

Since  $\langle \tilde{e}^n(t), \tilde{e}^n(t) \rangle^t = 1$ , we have

$$(6.4) \quad \ddot{\tilde{e}}^i(t) = - \sum_{j=1}^{n-1} \left( \tilde{\mathcal{R}}_{ij}(t) - \Omega_{ni}(t)\Omega_{nj}(t) \right) \tilde{e}^j(t) + \Omega_{ni}(t)\dot{\tilde{e}}^n(t).$$

Let  $U$  be a solution of the equation  $\dot{U}(t) = \frac{1}{2}U(t)\Omega(t)$  with  $U(0) = I$  and let  $E(t) = U(t)\tilde{E}(t)$ . By Lemma 3.4,  $E(t)$  and  $F(t) := \dot{E}(t)$  together form a Darboux basis. If we differentiate this twice, we obtain

$$\begin{aligned} & -\mathcal{R}(t)U(t)\tilde{E}(t) \\ &= -\mathcal{R}(t)E(t) \\ &= \ddot{E}(t) \\ &= \ddot{U}(t)\tilde{E}(t) + U(t)\Omega(t)\dot{\tilde{E}}(t) + U(t)\ddot{\tilde{E}}(t) \\ &= \ddot{U}(t)\tilde{E}(t) + U(t)\Omega(t)\dot{\tilde{E}}(t) + U(t)\ddot{\tilde{E}}(t) \end{aligned}$$

Since  $\Omega$  is a skew-symmetric matrix satisfying  $\Omega_{ij} = 0$  if  $i \neq n$  and  $j \neq n$ . It follows that

$$U(t)^T \mathcal{R}(t) U(t) = \begin{pmatrix} \tilde{\mathcal{R}}(t) - \frac{3}{4}\bar{\Omega} \otimes \bar{\Omega} & \frac{1}{2}\dot{\bar{\Omega}} \\ \frac{1}{2}\dot{\bar{\Omega}}^T & \frac{1}{4}|\bar{\Omega}|^2 \end{pmatrix},$$

where  $\bar{\Omega}$  is the vector in  $\mathbb{R}^n$  with  $i$ -th entry equal to  $\Omega_{ni}$ .  $\square$

Next, we define the reduced curvature operator of the Hamiltonian vector field  $\vec{H}$ . The Hamiltonian vector field  $\vec{H}$  is transversal to the distribution  $\Lambda$ . It follows that  $\vec{H}$  is not contained in any  $J_\alpha(t)$  for each  $t$ . Therefore, the reduction  $\tilde{J}_\alpha$  of the curve  $J_\alpha$  is defined by

$$\tilde{J}_\alpha := J_\alpha \cap \vec{H}^\perp / \mathbb{R}\vec{H}.$$

Note that  $\vec{H}_\alpha^\perp = \ker dH_\alpha = (T\Sigma_c)_\alpha$  if  $c$  is regular value of  $H$  and  $\alpha$  is in  $\Sigma_c := H^{-1}(c)$ .

We also let  $\tilde{\Lambda}_\alpha$  be the reduced distribution

$$\tilde{\Lambda}_\alpha := \Lambda_\alpha \cap \vec{H}^\perp / \mathbb{R}\vec{H}.$$

**Definition 6.2.** Assuming that the Hamiltonian vector field  $\vec{H}$  is regular. Let  $\tilde{R}_\alpha(t)$  be the curvature operators of the Jacobi curve  $\tilde{J}_\alpha(t)$  at  $\alpha$ . The *reduced curvature operator*  $\tilde{\mathfrak{R}} : \tilde{\Lambda} \rightarrow \tilde{\Lambda}$  of  $\vec{H}$  is defined by

$$\tilde{\mathfrak{R}}_\alpha = \tilde{R}_\alpha(0).$$

The following is an immediate consequence of Proposition 6.1.



**Proposition 6.3.** *Assume that the Hamiltonian vector field is monotone. For  $w$  in  $\Lambda \cap \vec{H}^\perp$ , we have*

$$\langle \tilde{\mathfrak{R}}(w), w \rangle = \langle \mathfrak{R}(w), w \rangle + \frac{3}{4}\omega([\vec{H}, [\vec{H}, \xi]], w)^2,$$

where  $\xi$  is a (local) section of  $(\Lambda \cap \vec{H}^\perp)^\perp$  and  $\perp$  denotes the orthogonal complement taken with respect to the canonical inner product of  $\vec{H}$ .

*Proof.* Using the notations of the proof of Proposition 6.1, we have  $\tilde{e}^n(t) = \varphi_t^* \xi(\alpha)$ . It follows that

$$\begin{aligned} \Omega_{ni}(0) &= -\omega(\ddot{e}^n(0), \tilde{e}^i(0)) \\ &= -\omega([\vec{H}, [\vec{H}, \xi]], \tilde{e}^i(0)). \end{aligned}$$

The rest follows from this.  $\square$

## 7. EXISTENCE OF INVARIANT DISTRIBUTIONS

Let  $\varphi_t$  be the flow of a regular Hamiltonian vector field  $\vec{H}$ . The point  $\varphi_t(\alpha)$  is a conjugate point of  $\alpha$  along the flow  $\varphi_t$  if  $d\varphi_t(\Lambda_\alpha)$  and  $\Lambda_{\varphi_t(\alpha)}$  do not intersect transversely. Equivalently,  $\varphi_t(\alpha)$  is a conjugate point if  $J_\alpha(0)$  and  $J_\alpha(t)$  do not intersect transversely.

In this section, we assume that a given Hamiltonian vector field  $\vec{H}$  is monotone and it does not contain any conjugate point. Under these assumptions, we show that there are always two Lagrangian distributions  $\Delta^\pm$  which are invariant under  $\varphi_t$ . Theorem 1.1 also follows from the following.

**Theorem 7.1.** *Assume that a given Hamiltonian vector field  $\vec{H}$  is monotone and its flow does not contain any conjugate point. Then the following holds:*

- (1)  $\Delta_\alpha^\pm := \lim_{t \rightarrow \pm\infty} J_\alpha(t)$  exists,
- (2)  $\Delta^\pm$  are Lagrangian distributions which are invariant under  $d\varphi_t$ ,
- (3)  $\Delta^+ \cap \Lambda = \Delta^- \cap \Lambda = \{0\}$ ,
- (4)  $\vec{H} \subseteq \Delta^+ \cap \Delta^-$ ,
- (5)  $\Delta^\pm \subseteq \vec{H}^\perp$ .

Note that the above theorem does not require any compactness assumption on  $\Sigma_c = H^{-1}(c)$ . For the proof of Theorem 7.1, it is convenient to introduce the reduction of  $d\varphi_t$  which is also needed in the later sections. Let us consider the quotient bundle  $\mathfrak{V} := \vec{H}^\perp / \mathbb{R}\vec{H}$ . Both the symplectic structure  $\omega$  and the flow  $d\varphi_t$  descend to  $\mathfrak{V}$ . The descended

objects are denoted by  $\tilde{\omega}$  and  $\widetilde{d\varphi_t}$ , respectively. The bundle  $\tilde{\Lambda}$  defined by  $\tilde{\Lambda} := (\Lambda \cap \vec{H}^\perp) / \mathbb{R}\vec{H}$  is a Lagrangian sub-bundle of  $\mathfrak{V}$ .

Let  $\tilde{J}_\alpha(t)$  be the reduced Jacobi curve defined by

$$\tilde{J}_\alpha(t) := \widetilde{d\varphi_t}^{-1}(\tilde{\Lambda}_{\varphi_t(\alpha)}) = J_\alpha(t) \cap \vec{H}_\alpha^\perp / \mathbb{R}\vec{H}.$$

Let  $(,)_\alpha^t$  be the canonical bilinear form (3.1) of the curve  $\tilde{J}_\alpha$ . Let  $(,)_\alpha := (,)_\alpha^t$ . This defines a bilinear form on  $\tilde{\Lambda}$ . Clearly,  $(,)$  is an inner product if  $<, >$  is. The canonical frames of  $\tilde{J}_\alpha$  are denoted by

$$\tilde{E}_\alpha(t) = (\tilde{e}_\alpha^1(t), \dots, \tilde{e}_\alpha^{n-1}(t))^T.$$

Next, we adopt an argument in [9] and prove the following result which holds true for a general regular curve in the Lagrangian Grassmannian.

**Proposition 7.2.** *Let  $J$  be a curve in the Lagrangian Grassmannian  $\mathcal{L}(\mathcal{V})$ . Let  $\Delta$  be a Lagrangian subspace of  $\mathcal{V}$  such that  $\Delta$  and  $J(t)$  intersect transversely for all  $t$ . Let  $v$  be a vector in  $\Delta$ . Assume that the curves  $J(t)$  and  $J(0)$  intersect transversely for all  $t$ . Then the same holds for the reduced curve  $\tilde{J}$ .*

*Proof.* Assume the contrary. Then there is a vector  $w$  in  $J(0) \cap v^\perp \cap (J(t_0) \oplus \mathbb{R}v)$ . Let  $E(t)$  be a canonical frame and let  $F(t) = \dot{E}(t)$ . Let  $D(t)$  be the matrix such that the components of

$$-\dot{D}(t)^T E(t) + D(t)^T F(t)$$

span  $\Delta$  and  $D(t)$  satisfies (4.3) with  $D(0) = I$ .

Let  $B$  be the matrix defined by

$$(7.1) \quad B(t) = D(t) \int_0^t D(s)^{-1} (D(s)^T)^{-1} ds$$

is a solution of (4.3) with initial conditions  $B(0) = 0$  and  $\dot{B}(0) = I$ .

Since  $w$  is contained in  $J(0)$ , we can let  $w = -a^T E(0)$  and get

$$w = a^T (-\dot{B}(t)^T E(t) + B(t)^T F(t)).$$

Since  $w$  is contained in  $J(t_0) \oplus \mathbb{R}v$  and  $v$  is transversal to the space  $J(t_0)$ , the  $J(t_0)$ -component of  $v$  is given by a multiple of the non-zero vector  $a^T B(t_0)^T F(t_0)$ . On the other hand, since  $v$  is contained in  $\Delta$ , there is a vector  $b$  such that

$$v = b^T (-\dot{D}(t)^T E(t) + D(t)^T F(t)).$$

It follows that  $B(t_0)a$  is a scalar multiple of  $D(t_0)b$ . Note that  $D(t_0)$  is invertible since  $\Delta$  and  $J(t_0)$  intersect transversely. Therefore, if we

combine the above considerations with (7.1), then  $a^T b > 0$ . However, since  $w$  is contained in  $v^\angle$ , we also have

$$\begin{aligned} 0 &= \omega(v, w) \\ &= -\omega(b^T F(0), a^T E(0)) \\ &= -b^T a. \end{aligned}$$

This gives a contradiction.  $\square$

*Proof of Theorem 7.1.* We prove the statements for  $\Delta^+$ . That of  $\Delta^-$  is similar and will be omitted. We will work with the reduced flow and find a Lagrangian sub-bundle  $\tilde{\Delta}^+$  in  $\mathfrak{V}$  which is invariant under  $\widetilde{d\varphi_t}$  instead. It follows that the distribution  $\Delta^+$  defined by

$$(7.2) \quad \Delta_\alpha^+ := \{v \in \vec{H}(\alpha)^\angle \mid v + \mathbb{R}\vec{H}(\alpha) \in \tilde{\Delta}_\alpha^+\}$$

is an invariant Lagrangian distribution.

Let  $\tilde{E}_\alpha(t) := (\tilde{e}_\alpha^1(t), \dots, \tilde{e}_\alpha^{n-1}(t))^T$  be a canonical frame of the reduced curve  $\tilde{J}_\alpha(t)$  at  $\alpha$  and let  $\tilde{F}_\alpha(t) = \dot{\tilde{E}}_\alpha(t)$ . Let  $B(s, t)$  be the matrices defined by

$$(7.3) \quad \tilde{E}_\alpha(t) = -B'(s, t)\tilde{E}_\alpha(s) + B(s, t)\tilde{F}_\alpha(s).$$

By differentiating (7.3) with respect to  $t$ , we obtain

$$(7.4) \quad \tilde{F}_\alpha(t) = -\dot{B}'(s, t)\tilde{E}_\alpha(s) + \dot{B}(s, t)\tilde{F}_\alpha(s)$$

and

$$\begin{aligned} & -\tilde{\mathcal{R}}_\alpha(t)B'(s, t)\tilde{E}_\alpha(s) + \tilde{R}_\alpha(t)B(s, t)\tilde{F}_\alpha(s) \\ &= \tilde{\mathcal{R}}_\alpha(t)\tilde{E}_\alpha(t) \\ &= \ddot{B}'(s, t)\tilde{E}_\alpha(s) - \ddot{B}(s, t)\tilde{F}_\alpha(s). \end{aligned}$$

It follows that

$$\ddot{B}(s, t) = -\tilde{\mathcal{R}}_\alpha(t)B(s, t).$$

Let  $U_\alpha(s, t) := \dot{B}(s, t)B(s, t)^{-1}$ . It satisfies

$$\dot{U}_\alpha(s, t) + (U_\alpha(s, t))^2 + \tilde{\mathcal{R}}_\alpha(t) = 0.$$

By assumption and Proposition 7.2,  $\tilde{\mathcal{R}}_\alpha(t)$  satisfies Assumption 4.3. It follows from Lemma 4.4 that  $U_\alpha^+(t) = \lim_{s \rightarrow \infty} U_\alpha(s, t)$  exists. Finally, we define

$$(7.5) \quad \tilde{\Delta}_\alpha^+ := \mathbf{span}\{\tilde{F}_\alpha(0) - U_\alpha^+(0)\tilde{E}_\alpha(0)\}.$$

If we set  $t = 0$  in (7.3) and (7.4), then we obtain

$$\tilde{J}_\alpha(s) = \mathbf{span}\{\tilde{F}_\alpha(0) - U_\alpha(s, 0)\tilde{E}_\alpha(0)\}.$$

Therefore, we have  $\lim_{s \rightarrow \infty} \tilde{J}_\alpha(s) = \tilde{\Delta}_\alpha^+$  and (1) follows. It also follows from (7.5) that  $\tilde{\Delta}^+ \cap \tilde{\Lambda} = \{\mathbb{R}\vec{H}\}$ . Since  $\vec{H}$  is not contained in  $\Lambda$ , (3) follows. (4) follows from (7.2) and (5) follows from taking skew-orthogonal complement in (4). Finally, by Proposition 5.5,

$$d\varphi_s(\tilde{J}_\alpha(t)) = \tilde{J}_{\varphi_s(\alpha)}(t - s).$$

If we let  $t \rightarrow \infty$ , then we see that  $\tilde{\Delta}^+$  is invariant under  $\widetilde{d\varphi_t}$ . Since  $\vec{H}$  is also invariant under  $d\varphi_t$ , (2) follows.  $\square$

## 8. RIGIDITY OF THE REDUCED CURVATURE

In this section, we will give the proof of Theorem 1.2. In fact, Theorem 1.2 is an immediate consequence of the following result.

**Theorem 8.1.** *Let  $c$  be a regular value of the Hamiltonian  $H$  and let  $\Sigma_c := H^{-1}(c)$ . Assume that the Hamiltonian vector field  $\vec{H}$  is regular and the differential  $d\varphi_t$  of its flow  $\varphi_t$  preserves a Lagrangian distribution on  $\Sigma_c$ . Then the trace  $\tilde{\mathfrak{r}}$  of the reduced curvature  $\tilde{\mathfrak{R}}$  satisfies*

$$\int_{\Sigma_c} \tilde{\mathfrak{r}}_\alpha d\mu(\alpha) \leq 0,$$

where  $\mu$  is any invariant measure defined on  $\Sigma_c$ . Moreover, equality holds only if  $\tilde{\mathfrak{r}} = 0$  on the support of  $\mu$ .

*Proof.* Let  $\tilde{\Delta}$  be defined by  $\tilde{\Delta} := \Delta \cap \vec{H}^\perp / \mathbb{R}\vec{H}$ . Then  $\tilde{\Delta}$  is a sub-bundle of  $\mathfrak{V}$  which is invariant under  $\widetilde{d\varphi_t}$ . Let  $E_\alpha(t) = (e_\alpha^1(t), \dots, e_\alpha^{n-1}(t))^T$  be a canonical frame at  $\alpha$ , let  $F_\alpha(t) = \dot{E}_\alpha(t)$ , and let  $S_0$  be the matrix such that

$$F_\alpha(0) + S_0 E_\alpha(0)$$

span the space  $\tilde{\Delta}_\alpha$ .

It follows that

$$F_\alpha(0) - S_0 E_\alpha(0) = B_\alpha(t)^T F_\alpha(t) - \dot{B}_\alpha(t)^T E_\alpha(t)$$

where  $B_\alpha(t)$  is a solution of (4.3) satisfying the initial conditions  $B_\alpha(0) = I$  and  $\dot{B}_\alpha(0) = S_0$  and  $\mathcal{R} = \tilde{\mathcal{R}}_\alpha$  is the curvature of the reduced Jacobi curve  $\tilde{J}_\alpha(t)$ .

Let  $S_\alpha(t) = \dot{B}_\alpha(t) B_\alpha(t)^{-1}$ . Then  $S_\alpha$  is the solution of (4.4) which satisfies the initial condition  $S_\alpha(0) = S_0$ . It follows that the trace  $\mathbf{tr}(S_\alpha(t))$  of  $S_\alpha(t)$  satisfies the following equation

$$(8.1) \quad \mathbf{tr}(\dot{S}_\alpha(t)) + \mathbf{tr}(S_\alpha(t)^2) + \mathbf{tr}(\tilde{\mathcal{R}}_\alpha(t)) = 0.$$

Since  $\mathbf{tr}(S_\alpha(t))$  is independent of the choice of frames  $\tilde{E}_\alpha(t)$ , it defines a function  $\alpha \mapsto \mathbf{tr}(S_\alpha(t))$ . Moreover, we have  $\mathbf{tr}(S_\alpha(t)) = \mathbf{tr}(S_{\varphi_t(\alpha)}(0))$ .

Therefore, if we integrate (8.1) with respect to the invariant measure  $\mu$ , then we obtain

$$\begin{aligned} 0 &= -\frac{d}{dt} \int_M \mathbf{tr}(S_{\varphi_t(\alpha)}(0)) d\mu(\alpha) \\ &= \int_M \mathbf{tr}(S_{\varphi_t(\alpha)}(0)^2) + \tilde{\mathbf{r}}_{\varphi_t(\alpha)} d\mu(\alpha) \end{aligned}$$

Since  $\mu$  is an invariant measure of  $\varphi_t$ , it follows that  $\int_M \tilde{\mathbf{r}}_\alpha d\mu(\alpha) \leq 0$ . Moreover, equality holds only if  $S_\alpha(t) = 0$  for  $\mu$ -almost all  $\alpha$ . Since  $t \mapsto S_\alpha(t)$  is smooth, there is a set of full measure  $\mathcal{O}$  in  $M$  such that  $S_\alpha(t) = 0$  for all  $t$  and for each  $\alpha$  in  $\mathcal{O}$ .

Finally, it follows from (8.1) and the smoothness of  $\tilde{\mathbf{r}}$  that  $\tilde{\mathbf{r}} = 0$  on the support of  $\mu$ .  $\square$

## 9. HYPERBOLICITY UNDER NEGATIVE REDUCED CURVATURE

In this section, we show that the Hamiltonian flow of a monotone Hamiltonian vector field is Anosov if the reduced curvature is bounded above and below by negative constants. First, we show that if the reduced curvature is everywhere non-positive, then the Hamiltonian flow has no conjugate point. A proof of this can be found in [21]. We supply the proof here for completeness.

**Theorem 9.1.** *Assume that the reduced curvature of a regular Hamiltonian vector field  $\vec{H}$  is non-positive. Then, for each  $\tilde{w}$  in  $\tilde{\Lambda}$ ,  $|\widetilde{d\varphi_t}(\tilde{w})^h|$  is increasing for all  $t > 0$  and decreasing for all  $t < 0$ . In particular, the flow of  $\vec{H}$  has no conjugate point.*

*Proof.* We will only do the case  $t > 0$ . Let  $\tilde{E}_\alpha(t) = (\tilde{e}_\alpha^1(t), \dots, \tilde{e}_\alpha^{n-1}(t))^T$  be a canonical frame of the Jacobi curve at  $\alpha$ . Let  $B(t)$  be the solution of (4.3) with initial conditions  $B(0) = 0$  and  $\dot{B}(0) = I$ . Then

$$\tilde{E}_\alpha(0) = \dot{B}(t)^T \tilde{E}_\alpha(t) - B(t)^T \tilde{F}_\alpha(t).$$

In other words, if we define  $S(t) = \dot{B}(t)B(t)^{-1}$ , then  $S(t)$  is a solution of the matrix Riccati equation (4.4) which is defined for all  $t \neq 0$ . Since  $\frac{1}{t}I$  is also a solution of (4.4) with  $R(t) \equiv 0$ , it follows from Theorem 4.2 that  $S(t) \geq \frac{1}{t}I$ . It also follows from Theorem 4.2 that  $S(t)$  is bounded above by the solutions of the equation

$$\dot{S}(t) + \mathcal{R}_\alpha(t) = 0.$$

It follows that  $S(t)$  is defined for all  $t$  and  $B(t)$  is invertible. Therefore, by Proposition 7.2, there is no point conjugate to  $\alpha$  along  $\varphi_t$ .

Therefore,

$$\begin{aligned} \frac{d}{dt} |(\widetilde{d\varphi_t}(\tilde{w}))^h|^2 &= 2b^T B(t)^T \dot{B}(t)b \\ &= 2b^T B(t)^T S(t)B(t)b > 0 \end{aligned}$$

for all  $t > 0$ .  $\square$

**Theorem 9.2.** *Assume that there are positive constants  $k$  and  $K$  such that the reduced curvature  $\tilde{\mathfrak{R}}$  satisfies  $-K^2 I \geq \tilde{\mathfrak{R}} \geq -k^2 I$  on  $\Sigma_c := H^{-1}(c)$ . Then there is a Riemannian inner product and invariant distributions  $\Delta^s$  and  $\Delta^u$  defined on  $\bigcup_{\alpha \in \Sigma_c} \vec{H}^\angle(\alpha)$  satisfying the followings:*

- (1)  $\vec{H}^\angle = \mathbf{span}\{\vec{H}\} \oplus \Delta^u \oplus \Delta^s$ ,
- (2)  $\Delta^+ = \mathbf{span}\{\vec{H}\} \oplus \Delta^s$ ,
- (3)  $\Delta^- = \mathbf{span}\{\vec{H}\} \oplus \Delta^u$ ,
- (4) *there is a constant  $C > 0$  such that  $|d\varphi_t(w)| \leq Ce^{-Kt}|w|$  for all  $t \geq 0$  and for all  $w$  in  $\Delta^s$ ,*
- (5)  *$|d\varphi_{-t}(w)| \leq Ce^{-Kt}|w|$  for all  $t \geq 0$  and for all  $w$  in  $\Delta^u$ .*

*In particular, if  $c$  is a regular value of  $H$ , then the flow  $\varphi_t$  is Anosov on  $\Sigma_c$ .*

*Proof.* We use the notations in the proof of Theorem 7.1. Let  $\dot{D}^+(t) = U^+(t)D^+(t)$  with  $D^+(0) = I$ . If  $\tilde{w}$  be a vector in  $\tilde{\Delta}^+$ . Then there is a vector  $b$  such that

$$\tilde{w} = b^T(-\dot{D}^+(t)^T \tilde{E}_\alpha(t) + D^+(t)^T \tilde{F}_\alpha(t)).$$

We extend the canonical inner product defined on  $\tilde{\Lambda}$  to an inner product, still denoted by  $\langle \cdot, \cdot \rangle$ , of the bundle  $\mathfrak{V}$  such that the basis  $\tilde{e}_\alpha^1(0), \dots, \tilde{e}_\alpha^{n-1}(0), \tilde{f}_\alpha^1(0), \dots, \tilde{f}_\alpha^{n-1}(0)$  is orthonormal. It follows that

$$(9.1) \quad |\widetilde{d\varphi_t}(\tilde{w})|^2 = |D^+(t)b|^2 + |U^+(t)D^+(t)b|^2.$$

By Lemma 4.6, we have

$$U^+ \leq -kI \quad \text{and} \quad |D^+(t)b|^2 \leq |b|^2 e^{-2Kt}.$$

By combining this with (9.1), we obtain

$$\begin{aligned} |\widetilde{d\varphi_t}(\tilde{w})|^2 &\leq (1 + k^2) |D^+(t)b|^2 \\ &\leq (1 + k^2) |b|^2 e^{-2Kt} \\ &\leq \frac{1 + k^2}{1 + K^2} |\tilde{w}|^2 e^{-2Kt}. \end{aligned}$$

The rest follows from [22, Proposition 5.1] and the definition of  $\Delta^+$  in the proof of Theorem 7.1.  $\square$

## 10. ON THE INVARIANT BUNDLES OF THE REDUCED FLOW

Let  $\tilde{J}_\alpha$  be the reduced Jacobi curve of  $J_\alpha$ . The reduced Jacobi curve and the derivative curve  $\tilde{J}_\alpha^o$  give a splitting of the bundle  $\mathfrak{V} = \tilde{J}_\alpha(0) \oplus \tilde{J}_\alpha^o(0)$ . Let  $\tilde{v}$  be an element in  $\mathfrak{V}$ . The  $\tilde{J}_\alpha(0)$ - and the  $\tilde{J}_\alpha^o(0)$ -components of  $\tilde{v}$  are denoted by  $\tilde{v}^v$  and  $\tilde{v}^h$  respectively.

In this section, we prove the following characterization of the invariant bundles  $\tilde{\Lambda}^\pm$  defined in the proof of Theorem 7.1.

**Theorem 10.1.** *Assume that the Hamiltonian vector field  $\vec{H}$  is monotone. Assume that  $\Sigma_c := H^{-1}(c)$  is compact and the flow of  $\vec{H}$  has no conjugate point on  $\Sigma_c$ . Suppose that there is no vector  $\tilde{w}$  in  $\mathfrak{V}$  such that  $|\widetilde{d\varphi_t}(\tilde{w})^h|$  is bounded for all  $t$ . Then*

$$\tilde{\Delta}^\pm = \left\{ \tilde{w} \mid \sup_{\pm t \geq 0} |\widetilde{d\varphi_t}(\tilde{w})^h| < +\infty \right\}.$$

In particular, the above theorem applies when the flow of  $\vec{H}$  is Anosov on  $\Sigma_c$ .

**Lemma 10.2.** *Assume that the reduced curvature  $\tilde{\mathfrak{R}}$  of a monotone Hamiltonian vector field satisfies  $\tilde{\mathfrak{R}} \geq -k^2 I$ . Let  $\tilde{v}$  be in  $\mathfrak{V}$  such that  $|\widetilde{d\varphi_t}(\tilde{v})^h|$  is uniformly bounded for all  $t > 0$  (resp.  $t < 0$ ). Then  $\tilde{v}$  is contained in  $\tilde{\Delta}^+$  (resp.  $\tilde{\Delta}^-$ ).*

*Proof.* We will only prove the statement for  $\tilde{\Delta}^+$ . The one for  $\tilde{\Delta}^-$ , being very similar, will be omitted. Let  $\tilde{v}$  be a tangent vector in  $T_\alpha M$  such that  $t \mapsto \widetilde{d\varphi_t}(\tilde{v})$  is uniformly bounded for all  $t > 0$ . Let  $\tilde{v}_t$  be a vector in  $J_\alpha(t)$  such that the horizontal components of  $\tilde{v}$  and  $\tilde{v}_t$  are the same. It follows that  $\tilde{v} - \tilde{v}_t$  is vertical for each  $t$ .

Let  $\tilde{E}(t) = (\tilde{e}_\alpha^1(t), \dots, \tilde{e}_\alpha^{n-1}(t))^T$  be canonical frame and let  $\tilde{F}(t) = \dot{\tilde{E}}(t)$ . Let  $B(s)$  be the solution of (4.3) with initial conditions  $B(0) = 0$  and  $B'(0) = I$ . Let  $b(t)$  be a family of vectors in  $\mathbb{R}^n$  defined by  $\tilde{v} - \tilde{v}_t = b(t)^T \tilde{E}(0)$ . Then we have

$$\tilde{v} - \tilde{v}_t = b(t)^T B'(s)^T \tilde{E}(s) - b(t)^T B(s)^T \tilde{F}(s).$$

By assumption, there is a constant  $K > 0$  such that  $|B(t)b(t)| \leq K$ . By Lemma 4.7, there is  $T_n > 0$  such that

$$\frac{K}{|b(t)|} \geq \frac{|B(t)b(t)|}{|b(t)|} \geq n$$

for all  $t > T_n$ .

Therefore,  $\lim_{t \rightarrow \infty} b(t) = 0$  and  $\lim_{t \rightarrow \infty} \tilde{v}_t = \tilde{v}$ . Since  $\tilde{v}_t$  is contained in  $\tilde{J}_\alpha(t)$  for all  $t > 0$ ,  $\tilde{v}$  is contained in  $\tilde{\Delta}_\alpha^+$  as claimed.  $\square$

**Lemma 10.3.** *Suppose that the assumptions of Theorem 10.1 are satisfied. Then for each  $s_0 > 0$  (resp.  $s_0 < 0$ ), there is a constant  $C > 0$  such that*

$$|\widetilde{d\varphi}_t(\tilde{w})^h| \geq C|\widetilde{d\varphi}_s(\tilde{w})^h|$$

for all  $\tilde{w}$  in  $\tilde{\Lambda}$  and for all  $t \geq s \geq s_0$  (resp.  $t \leq s \leq s_0$ ).

*Proof.* Suppose that the conclusion does not hold. Then there are vectors  $\tilde{w}_n$  in  $\tilde{\Lambda}$  and numbers  $t_n \geq s_n \geq s_0$  such that

$$|\widetilde{d\varphi}_{t_n}(\tilde{w}_n)^h| < \frac{1}{n}|\widetilde{d\varphi}_{s_n}(\tilde{w}_n)^h|.$$

By multiplying  $\tilde{w}_n$  by a constant, we can assume that  $|\tilde{w}_n| = 1$ . By compactness, we can assume that  $\tilde{w}_n$  converges to  $\tilde{w}$  in  $\tilde{\Lambda}$ . Let  $u_n$  be the number which achieves the maximum of  $|\widetilde{d\varphi}_t(\tilde{w}_n)^h|$  over  $t$  in  $[0, t_n]$ . It follows that

$$|\widetilde{d\varphi}_{u_n}(\tilde{w}_n)^h| \geq |\widetilde{d\varphi}_{s_0}(\tilde{w}_n)^h|$$

is bounded below by a positive constant uniformly in  $n$  since  $\tilde{w}_n$  is convergent. Therefore,  $u_n$  is also bounded below by a positive constant uniformly in  $n$ .

Let  $\tilde{v}_n = \frac{\widetilde{d\varphi}_{u_n}(\tilde{w}_n)}{|\widetilde{d\varphi}_{u_n}(\tilde{w}_n)^h|}$  and let  $a_n$  be vectors defined by

$$\tilde{w}_n = a_n^T (\dot{B}(t)^T E(t) - B(t)^T F(t))$$

where  $B$  is a solution of (4.3) with initial conditions  $B(0) = 0$  and  $\dot{B}(0) = I$ .

Let  $S(t) = B(t)^{-1} \dot{B}(t)$ . Then  $S(t)$  satisfies (4.4). By Lemma 4.5, it follows that  $\tilde{v}_n$  satisfies

$$\begin{aligned} |\tilde{v}_n| &\leq 1 + \frac{|\widetilde{d\varphi}_{u_n}(\tilde{w}_n)|}{|\widetilde{d\varphi}_{u_n}(\tilde{w}_n)^h|} \\ &= 1 + \frac{|\dot{B}(u_n)a_n|}{|B(u_n)a_n|} \\ &\leq 1 + k \coth(ku_n). \end{aligned}$$

Since  $u_n$  is bounded uniformly from below by a positive constant,  $|\tilde{v}_n|$  is also bounded uniformly and we can assume that  $\tilde{v}_n$  converges to a vector  $\tilde{v}$ . By the definition of  $u_n$ , we have

$$(10.1) \quad |\widetilde{d\varphi}_t(\tilde{v}_n)^h| \leq 1$$

for  $-u_n \leq t \leq t_n - u_n$ . By assumption,  $\widetilde{d\varphi}_{-u_n}(\tilde{v}_n)$  is contained in  $\tilde{\Lambda}$  and  $|\widetilde{d\varphi}_{t_n - u_n}(\tilde{v}_n)^h| < \frac{1}{n}$ . If both  $u_n$  and  $t_n - u_n$  have convergent subsequence, then it violates the assumption that there is no conjugate



point. If both  $-u_n \rightarrow -\infty$  and  $t_n - u_n \rightarrow +\infty$ . Then this violates the assumption that there is no bounded reduced non-zero Jacobi field. If one of  $-u_n$  or  $t_n - u_n$  has a convergent subsequence, then one of  $\widetilde{d\varphi}_{-u_n}(\tilde{v}_n)$  or  $\widetilde{d\varphi}_{t_n - u_n}(\tilde{v}_n)$  converges to a vector in  $\tilde{\Lambda}$ . This vector is also contained in either  $\tilde{\Delta}^+$  or  $\tilde{\Delta}^-$  by Lemma 10.2 and (10.1). This violates (3) of Theorem 7.1.  $\square$

*Proof of Theorem 10.1.* One inclusion follows from Lemma 10.2. For the other inclusion, let  $\tilde{w}$  be in  $\tilde{\Delta}_\alpha^+$ . Let  $\tilde{w}_\tau$  be the vector in  $\tilde{J}_\alpha(\tau)$  such that  $\tilde{w}^h = \tilde{w}_\tau^h$ . By the proof of Lemma 10.2, we have  $\lim_{\tau \rightarrow \infty} \tilde{w}_\tau = w$ . Fix  $s_0 < 0$ . By Lemma 10.3, there is a constant  $C > 0$  such that

$$|\widetilde{d\varphi}_t(\tilde{u})^h| \geq C|\widetilde{d\varphi}_s(\tilde{u})^h|$$

for all  $t \leq s \leq s_0$  and for all  $\tilde{u}$  in  $\tilde{\Lambda}$ .

Let  $\tilde{u} = \widetilde{d\varphi}_\tau(\tilde{w}_\tau)$ ,  $t = -\tau$ , and  $s = -\tau + \epsilon$ . Then we obtain

$$|\tilde{w}_\tau^h| \geq C|\widetilde{d\varphi}_\epsilon(\tilde{w}_\tau)^h|.$$

By letting  $\tau$  goes to  $+\infty$ , we obtain

$$(10.2) \quad |\tilde{w}^h| \geq C|\widetilde{d\varphi}_\epsilon(\tilde{w})^h|.$$

Therefore,  $|\widetilde{d\varphi}_\epsilon(\tilde{w})^h| < +\infty$  for all  $\epsilon \geq 0$ .  $\square$

## 11. MONOTONE ANOSOV HAMILTONIAN FLOWS WITHOUT CONJUGATE POINT

In this section, we give various equivalent conditions which guarantee that a monotone Hamiltonian vector field without conjugate point is Anosov. More precisely, we will prove the following.

**Theorem 11.1.** *Let  $\vec{H}$  be a monotone Hamiltonian vector field without conjugate point. Assume that  $\Sigma_c = H^{-1}(c)$  is compact. Then the followings are equivalent.*

- (1)  $\tilde{\Delta}^+ \cap \tilde{\Delta}^- = \{0\}$ ,
- (2)  $\tilde{\Lambda} = \tilde{\Delta}^+ \oplus \tilde{\Delta}^-$ ,
- (3) *there is no vector  $\tilde{w}$  in  $\mathfrak{V}$  such that  $|\widetilde{d\varphi}_t(\tilde{w})^h|$  is bounded uniformly in  $t$ ,*
- (4) *there are constants  $c_1, c_2 > 0$  such that*

$$|\widetilde{d\varphi}_{\pm t}(\tilde{w})| \leq c_1 |\tilde{w}| e^{-c_2 t}$$

*for all  $t \geq 0$  and  $\tilde{w}$  in  $\tilde{\Delta}^\pm$ .*

**Lemma 11.2.** *Under the assumptions of Theorem 10.1,*

$$\lim_{t \rightarrow \pm\infty} \sup_{|\tilde{w}|=1, \tilde{w} \in \tilde{\Delta}^\pm} |\widetilde{d\varphi}_t(\tilde{w})| = 0.$$

*Proof.* Suppose the statement for  $\tilde{\Delta}^+$  does not hold. Then there is  $\epsilon > 0$ , a sequence  $t_n > 0$  going to  $\infty$ , and a sequence  $\tilde{w}_n$  in  $\tilde{\Delta}^+$  satisfying  $|\tilde{w}_n| = 1$  such that

$$|\widetilde{d\varphi}_{t_n}(\tilde{w}_n)| > \epsilon.$$

Since  $\widetilde{d\varphi}_{t_n}(\tilde{w}_n)$  is contained in  $\tilde{\Delta}^+$ ,  $|\widetilde{d\varphi}_{t_n}(\tilde{w}_n)|$  is uniformly bounded in  $n$  by compactness and the proof of Theorem 10.1. Therefore,  $\widetilde{d\varphi}_{t_n}(\tilde{w}_n)$  converges to  $\tilde{w} \neq 0$ . Since  $\widetilde{d\varphi}_{t_n}(\tilde{w}_n)$  is contained in  $\tilde{\Delta}$ ,  $|\widetilde{d\varphi}_{t+t_n}(\tilde{w}_n)|$  is uniformly bounded for all  $n$  and  $t \geq -t_n$  by (10.2). Hence, by letting  $n \rightarrow \infty$ ,  $|\widetilde{d\varphi}_t(\tilde{w})|$  is uniformly bounded in  $t$ . This contradicts the assumption of the lemma.  $\square$

**Lemma 11.3.** *Let  $\vec{H}$  be monotone and without conjugate point. Let  $c$  be a regular value of  $H$  and assume that  $\Sigma_c = H^{-1}(c)$  is compact. Then there is no vector  $\tilde{w}$  in  $\mathfrak{V}$  such that  $|\widetilde{d\varphi}_t(\tilde{w})^h|$  is bounded for all  $t$  if and only if there are constants  $c_1, c_2 > 0$  such that*

$$(11.1) \quad |\widetilde{d\varphi}_{\pm t}(\tilde{w})| \leq c_1 |\tilde{w}| e^{-c_2 t}$$

for all  $t \geq 0$  and  $\tilde{w}$  in  $\tilde{\Delta}^\pm$ .

*Proof.* Clearly, (11.1) implies that  $|\widetilde{d\varphi}_t(\tilde{w})^h|$  is not bounded for all  $t$ . Conversely, let

$$\phi^+(t) = \sup_{|\tilde{w}|=1, \tilde{w} \in \tilde{\Delta}^+} |\widetilde{d\varphi}_t(\tilde{w})|.$$

Then  $\phi^+$  is uniformly bounded for all  $t \geq 0$  (see (10.2)),  $\phi^+(t+s) \leq \phi^+(s)\phi^+(t)$  for all  $s, t \geq 0$ , and  $\lim_{s \rightarrow \infty} \phi^+(s) = 0$  (Lemma 11.2). The rest follows from [10, Lemma 3.12].  $\square$

*Proof of Theorem 11.1.* By a count in dimensions, (1) and (2) are equivalent. By Lemma 10.2, (1) implies (3). By Theorem 10.1, (3) implies (1). (3) and (4) are equivalent by Lemma 11.3.  $\square$

*Proof of Theorem 1.4.* By Theorem 11.1, it is enough to show that (3) of Theorem 11.1 is equivalent to (1) of Theorem 1.4. This, in turn, follows from [22, Proposition 5.1].  $\square$

## 12. THE CASE WITH NON-POSITIVE REDUCED CURVATURE

In this section, we give the proof of Theorem 1.5. Under the assumption that the reduced curvature of  $\vec{H}$  is non-positive, the following is a characterization of when the flow of  $\vec{H}$  is Anosov.

**Lemma 12.1.** *Assume that, for each  $\tilde{v}$  in  $\tilde{\Lambda}$ ,  $|\widetilde{d\varphi_t}(\tilde{v})^h|$  is increasing for each  $t > 0$  and decreasing for each  $t < 0$ . Then the followings are equivalent.*

- (1) *the Hamiltonian flow is Anosov,*
- (2)  $\bigcap_{t \in \mathbb{R}} \tilde{J}^o(t) = \emptyset.$

*In particular, the above conditions are equivalent if the reduced curvature of the Hamiltonian is non-positive.*

*Proof.* Let us fix a vector  $\tilde{w}$  and let  $b(t)$  be defined by

$$\tilde{w} = -\dot{b}(t)^T E(t) + b(t)^T F(t).$$

First, assume that  $\tilde{w}$  is contained in  $\bigcap_{t \in \mathbb{R}} \tilde{J}^o(t)$ . By assumption, we have  $\dot{b} \equiv 0$ . Therefore,  $b(t)$  is constant independent of  $t$ . It follows that  $|\widetilde{d\varphi_t}(\tilde{w})^h|$  is constant and the Hamiltonian flow is not Anosov by Theorem 11.1.

Conversely, by assumption and (10.2),  $|\widetilde{d\varphi_{\pm t}}(\tilde{w})^h| \leq |\tilde{w}^h|$  for all  $t \geq 0$  and for all  $\tilde{w}$  in  $\tilde{\Delta}^\pm$ . Since  $\tilde{\Delta}^\pm$  is invariant, we have  $|\widetilde{d\varphi_{\pm t+s}}(\tilde{w})^h| \leq |\widetilde{d\varphi_s}(\tilde{w})^h|$  for all  $s$ . Therefore, if  $\tilde{w}$  is in  $\tilde{\Delta}^+ \cap \tilde{\Delta}^-$ , then it follows that  $t \mapsto |\widetilde{d\varphi_{\pm t}}(\tilde{w})^h|$  is both non-increasing and non-decreasing. Therefore,  $t \mapsto |\widetilde{d\varphi_t}(\tilde{w})^h|$  is constant in  $t$ .

Let  $U^+$  be as in Theorem 7.1 and let  $D^+$  be defined by  $\dot{D}^+(t) = U^+(t)D^+(t)$  with initial condition  $D^+(0) = I$ . It follows that

$$0 \geq \frac{d}{dt} |D^+(t)\tilde{b}|^2 = 2 \left\langle U^+(t)D^+(t)\tilde{b}, D^+(t)\tilde{b} \right\rangle$$

for all  $t > 0$  and for all vector  $\tilde{b}$ . Since  $D^+$  is invertible,  $U^+ \leq 0$ .

Let  $b$  be a vector in  $\mathbb{R}^n$  such that

$$\tilde{w} = b^T (D^+(t)^T F(t) - \dot{D}^+(t)^T E(t)).$$

It follows that  $|\widetilde{d\varphi_t}(\tilde{w})^h| = |D^+(t)b|$  is constant and we have

$$0 = \frac{1}{2} \frac{d}{dt} (b^T D^+(t)^T D^+(t)b) = b^T D^+(t)^T U^+(t) D^+(t)b.$$

Since  $U^+ \leq 0$ , we have  $\dot{D}^+(t)^T b = U^+(t)D^+(t)^T b = 0$ . Since  $D^+(t)^T b = b$ , we have  $\tilde{w} = b^T F(t)$ . This shows (2) implies (1).  $\square$

**Proposition 12.2.** *Assume that the Hamiltonian flow has no conjugate point. Fix a vector  $b$ . If  $\tilde{\mathcal{R}}_\alpha(t)b^T\tilde{E}(t) \geq 0$  for all  $t$ , then  $\tilde{\mathcal{R}}_\alpha(t)b^T\tilde{E}(t) = 0$  for all  $t$  and  $b^T\tilde{F}(0)$  is contained in  $\bigcap_{t \in \mathbb{R}} \tilde{J}^o(t)$ .*

*Proof.* Let  $u(t) = b^TU^+(t)b$ . Then

$$\dot{u}(t) + u(t)^2 + r(t) = 0$$

where  $r(t) = b^T\tilde{\mathcal{R}}_\alpha(t)b + b^TU^+(t)^2b - (b^TU^+(t)b)^2 \geq 0$ .

By an argument in [16], we see that  $u \equiv 0$ . Therefore,  $r \equiv 0$  and so  $b^T\tilde{\mathcal{R}}_\alpha(t)b \equiv 0$ . It also follows that  $U^+(t)b \equiv 0$  and hence  $\dot{U}^+(t)b \equiv 0$ . Therefore, by matrix Riccati equation of  $U^+$ , we have  $\tilde{\mathcal{R}}_\alpha(t)b \equiv 0$ . Finally, we have

$$\frac{d}{dt}b^TF(t) = -b^T\tilde{\mathcal{R}}_\alpha(t)E(t) = 0.$$

□

**Proposition 12.3.** *Assume that, for each  $\tilde{v}$  in  $\tilde{\Lambda}$ ,  $|(\widetilde{d\varphi_t}(\tilde{v}))^h|$  is increasing for each  $t > 0$  and is decreasing for each  $t < 0$ . If, for each  $\alpha$ ,  $\tilde{\mathcal{R}}_\alpha(t)b^T\tilde{E}(t) < 0$  for some  $t$ , then the Hamiltonian flow is Anosov.*

*Proof.* Suppose that the Hamiltonian flow is not Anosov. By Proposition 12.1, there is a vector  $b^TF(0) = b^TF(t)$  in  $J(t)$  for all  $t$ . If we differentiate this equation, then we obtain  $\tilde{\mathcal{R}}_\alpha(t)b^T\tilde{E}(t) \equiv 0$  which is a contradiction. □

### 13. ENTROPY ESTIMATES

In this section, we give the proofs of the two entropy estimates, Theorem 1.6 and 1.7. Let  $v$  be in  $\mathfrak{V}$ . The positive  $\chi^+$  and negative  $\chi^-$  Lyapunov exponents are defined by

$$\chi^\pm(v) = \lim_{t \rightarrow \pm\infty} \frac{1}{|t|} \log |\widetilde{d\varphi_t}(v)|.$$

Let  $E_\alpha^u$ ,  $E_\alpha^s$ , and  $E_\alpha^0$  be the subspaces of  $\mathfrak{V}$  defined by

$$\begin{aligned} E_\alpha^u &= \{v \in \mathfrak{V} | \chi^-(v) = -\chi^+(v) < 0\}, \\ E_\alpha^s &= \{v \in \mathfrak{V} | \chi^+(v) = -\chi^-(v) < 0\}, \\ E_\alpha^0 &= \{v \in \mathfrak{V} | \chi^-(v) = \chi^+(v) = 0\}. \end{aligned}$$

By Oseledets Theorem,  $\mathfrak{V}_\alpha = E_\alpha^u \oplus E_\alpha^s \oplus E_\alpha^0$  holds for  $\mu$ -almost all  $\alpha$ .

*Proof of Theorem 1.6.* The same argument as in [7, Proposition 2.1] shows that the skew orthogonal complement of  $E_\alpha^u$  is  $E_\alpha^u \oplus E_\alpha^0$ . If  $v$  is contained in  $E_\alpha^u$ , then  $|\widetilde{d\varphi_t}(v)|$  is bounded for all  $t \leq 0$ . By Lemma 10.2,  $v$  is contained in  $\tilde{\Delta}^-$ . Therefore,  $E_\alpha^u \subseteq \tilde{\Delta}_\alpha^- \subseteq E_\alpha^u \oplus E_\alpha^0$ .

By Pesin's formula [20],

$$h_\mu = \int_{\Sigma_c} \chi(\alpha) d\mu(\alpha),$$

where  $\chi(\alpha) = \lim_{t \rightarrow \infty} \frac{1}{|t|} \log |\det(\widetilde{d\varphi_t}|_{\tilde{\Delta}_\alpha^-})|$ . Here determinant is taken with respect to orthonormal frames of any Riemannian metric.

Let  $U(s, t)$  be as in the proof of Theorem 7.1 and let  $U^-(t) = \lim_{s \rightarrow -\infty} U(s, t)$ . Then  $\tilde{\Delta}^-$  is spanned by the components of

$$\tilde{F}_\alpha(0) - U_\alpha^-(0) \tilde{E}_\alpha(0) = B_\alpha(t)^T \tilde{F}_\alpha(t) - \dot{B}_\alpha(t)^T \tilde{E}_\alpha(t),$$

where  $B_\alpha(\cdot)$  is the solution of  $\ddot{B}_\alpha(t) = -\tilde{\mathcal{R}}_\alpha(t) B_\alpha(t)$  with  $B_\alpha(0) = I$ .

If we let  $\langle \cdot, \cdot \rangle$  be a Riemannian metric on  $\Sigma_c$  such that

$$\tilde{F}_\alpha(0) - U_\alpha^-(0) \tilde{E}_\alpha(0)$$

is orthonormal in  $\tilde{\Delta}_\alpha^-$ . If we use this Riemannian metric in the definition of  $\chi$ , then it follows that

$$\chi(\alpha) = \lim_{t \rightarrow \infty} \frac{1}{t} \log \det B_\alpha(t) = \lim_{t \rightarrow \infty} \frac{1}{t} \int_0^t \text{tr}(U_\alpha^-(s)) ds.$$

By Birkhoff's ergodic theorem and Pesin's formula, we have

$$h_\mu = \int_{\Sigma_c} \text{tr}(U_\alpha^-(0)) d\mu(\alpha).$$

By Cauchy-Schwarz's inequality, we obtain

$$h_\mu \leq (n-1)^{1/2} \left( \int_{\Sigma_c} (\text{tr } U_\alpha^-(0))^2 d\mu(\alpha) \right)^{1/2}.$$

Since  $U_\alpha^-(t) = U_{\tilde{d\varphi_t}(\alpha)}^-(0)$  and  $\mu$  is invariant, it follows from the matrix Riccati equation that

$$h_\mu \leq (n-1)^{1/2} \left( - \int_{\Sigma_c} \tilde{\mathfrak{r}}_\alpha d\mu(\alpha) \right)^{1/2}.$$

If equality holds, then  $U_\alpha^-(0)$  is constant for  $\mu$  almost all  $\alpha$ . It follows from the Riccati equation that  $\tilde{\mathfrak{R}}$  is constant on the support of  $\mu$ .  $\square$

*Proof of Theorem 1.7.* By [19, Lemma 3.1], we have

$$h_\mu \leq \liminf_{t \rightarrow 0} \frac{1}{t} \int_{\Sigma_c} \log(\text{ex}(\widetilde{d\varphi_t})) d\mu$$

where  $\mathbf{ex}\Phi$  is the expansion of the linear map  $\Phi$  defined as

$$\mathbf{ex}\Phi = \sup_S \det \Phi|_S$$

where the supremum is taken over all nontrivial subspaces  $S$ .

Let  $C(t)$  and  $D(t)$  be the matrices defined by

$$\tilde{E}_\alpha(0) = -\dot{C}(t)\tilde{E}_\alpha(t) + C(t)\tilde{F}_\alpha(t), \quad \tilde{F}_\alpha(0) = -\dot{D}(t)\tilde{E}_\alpha(t) + D(t)\tilde{F}_\alpha(t).$$

The matrices  $C(t)$  is a solution to the equation

$$(13.1) \quad \ddot{C}(t) = -\tilde{\mathcal{R}}_\alpha(t)C(t)$$

with initial conditions  $C(0) = 0$  and  $\dot{C}(0) = -I$ .

Similarly,  $D(t)$  is a solution of the same equation which satisfies  $D(0) = I$  and  $\dot{D}(0) = 0$ .

It follows that  $\widetilde{d\varphi_t}$  sends  $\tilde{E}_\alpha(0)$  and  $\tilde{F}_\alpha(0)$  to

$$\begin{pmatrix} -\dot{C}(t) & -\dot{D}(t) \\ C(t) & D(t) \end{pmatrix} \begin{pmatrix} \tilde{E}_{\varphi_t(\alpha)}(0) \\ \tilde{F}_{\varphi_t(\alpha)}(0) \end{pmatrix}.$$

Using (13.1), we see that

$$\begin{pmatrix} -\dot{C}(t) & -\dot{D}(t) \\ C(t) & D(t) \end{pmatrix} = \begin{pmatrix} I & 0 \\ 0 & I \end{pmatrix} + t \begin{pmatrix} 0 & \tilde{\mathcal{R}}(0) \\ -I & 0 \end{pmatrix} + o(t)$$

as  $t \rightarrow 0$ .

It follows as in [19] that

$$\mathbf{ex}(\widetilde{d\varphi_t}) = 1 + \frac{t}{2} \sum_{i=1}^{n-1} |\lambda_i - 1| + o(t),$$

where  $\lambda_i$  are eigenvalues of the matrix  $\tilde{\mathfrak{R}}$ . □

## REFERENCES

- [1] A. A. Agrachev, R. Gamkrelidze: Feedback-invariant optimal control theory and differential geometry, I. Regular extremals. J. Dynamical and Control Systems, 1997, v.3, 343–389.
- [2] A. A. Agrachev: Geometry of Optimal Control Problems and Hamiltonian Systems, Lecture Notes, 2004.
- [3] A. A. Agrachev: The curvature and hyperbolicity of Hamiltonian systems. Proceed. Steklov Math. Inst., 2007, v.256, 26–46.
- [4] A. A. Agrachev, N. Chtcherbakova, I. Zelenko: On curvatures and focal points of dynamical Lagrangian distributions and their reductions by first integrals. J. J. Dynamical and Control Systems, 2005, v.11, 297–327.
- [5] A. A. Agrachev, R. Gamkrelidze: Feedback-invariant optimal control theory and differential geometry, I. Regular extremals, J. Dynamical and Control Systems, v.3, 343–389, 1997.

- [6] D. V. Anosov: Geodesic flows on closed Riemannian manifolds of negative curvature. (Russian) Trudy Mat. Inst. Steklov. 90 1967.
- [7] W. Ballmann, M. P. Wojtkowski: An estimate for the measure theoretic entropy of geodesic flows. Ergodic Theory Dynam. Systems 9 (1989), 271-279.
- [8] F. C. Chittaro: An estimate for the entropy of Hamiltonian flows. Journal of Dynamical and Control Systems, Vol. 13, No. 1, 2007, 55-67.
- [9] G. Contreras, R. Iturriaga: Convex Hamiltonians without conjugate points. Ergodic Theory Dynam. Systems 19 (1999), no. 4, 901-952.
- [10] P. Eberlein: When is a geodesic flow of Anosov type? I. J. Differential Geometry 8 (1973), 437-463.
- [11] A. Freire, R. Mañé: On the entropy of the geodesic flow in manifolds without conjugate points. Invent. Math. 69 (1982), 375-392.
- [12] P. Foulon: Estimation de l'entropie des systèmes langrangiens sans points conjugués. Ann. Inst. Henri Poincaré 57 (1992), 117-146.
- [13] N. Innami: Natural Lagrangian systems without conjugate points. Ergodic Theory Dynam. Systems 14 (1994), no. 1, 169-180.
- [14] L. W. Green: A theorem of E. Hopf. Michigan Math. J. 5 (1958) 31-34.
- [15] M. W. Hirsch, C. C. Pugh, M. Shub: Invariant manifolds. Lecture Notes in Mathematics, Vol. 583. Springer-Verlag, Berlin-New York, 1977
- [16] E. Hopf: Closed surfaces without conjugate points. Proc. Nat. Acad. Sci. U. S. A. 34, (1948). 47-51.
- [17] C. B. Li, I. Zelenko: Differential geometry of curves in Lagrange Grassmannians with given young diagram, Differential Geom. Appl. 27 (2009), no. 6, 723-742.
- [18] R. Osserman, P. Sarnak: A new curvature invariant and entropy of geodesic flows. Invent. Math. 77 (1984), 455-462.
- [19] G. P. Paternain, J. Petean: The pressure of Ricci curvature.
- [20] Y. B. Pesin: Formulas for the entropy of a geodesic flow on a compact Riemannian manifold without conjugate point. Math. Notes 24 (1978), 796-805.
- [21] H. L. Royden: Comparison theorems for the matrix Riccati equation. Comm. Pure Appl. Math. 41 (1988), no. 5, 739-746.
- [22] M. P. Wojtkowski: Magnetic flows and Gaussian thermostats on manifolds of negative curvature. Fund. Math. 163 (2000), no. 2, 177-191.

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